$H^2\left(\mathbf{SL_2}\left(\mathbb{Z}[t,t^{-1}]\right);\mathbb{Q}\right)$ IS INFINITE-DIMENSIONAL

by

Sarah Christine Cobb

A dissertation submitted to the faculty of
The University of Utah
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

The University of Utah

May 2014

Copyright © Sarah Christine Cobb 2014

All Rights Reserved

The University of Utah Graduate School

STATEMENT OF DISSERTATION APPROVAL

The dissertation of **Sarah Christine Cobb**

has been approved by the following supervisory committee members:

Kevin Wortman	, Chair	**3/5/2014** Date Approved
Mladen Bestvina	, Member	**3/5/2014** Date Approved
Jonathan Chaika	, Member	**3/5/2104** Date Approved
Patrick Reynolds	, Member	**3/5/2014** Date Approved
Yair Glasner	, Member	**3/5/2014** Date Approved

and by **Peter Trapa** , Chair/Dean of

the Department/College/School of **Mathematics**

and by David B. Kieda, Dean of The Graduate School.

ABSTRACT

We construct a map from the 3-skeleton of the classifying space for $\Gamma = \mathbf{SL_2}(\mathbb{Z}[t, t^{-1}])$ to a Euclidean building on which Γ acts. We then find an infinite family of independent cocycles in the building and lift them to the classifying space, thus proving that the cohomology group $H^2\left(\mathbf{SL_2}\left(\mathbb{Z}[t, t^{-1}]\right); \mathbb{Q}\right)$ is infinite-dimensional.

CONTENTS

ACKNOWLEDGMENTS

I am more grateful than I can express to my advisor, Kevin Wortman, for his guidance, encouragement, and patience during the last five years. This work would not have been possible without him, and I am very glad to have had the chance to work with him.

I am thankful also to Brendan Kelly and Morgan Cesa for many, many conversations. They have given me explanations when I didn't understand mathematics, encouragement when I didn't like it, and friendship constantly.

I thank my dad, who taught me to love math when I was very young and who still occasionally helps me with my "homework"; I thank my mom who taught me about hard work and list-making.

Thanks to Kai-Uwe Bux for finding some errors in the manuscript; to my committee members Mladen Bestvina, Jon Chaika, Patrick Reynolds, and Yair Glasner; and to the numerous others who have shaped my mathematical understanding.

CHAPTER 1

INTRODUCTION

In [1], Krstić-McCool prove that $\Gamma = \mathbf{SL_2}(\mathbb{Z}[t,t^{-1}])$ is not F_2. In [2], Bux-Wortman use geometric methods to show that Γ is not FP_2. In particular, they use the action of $\mathbf{SL_2}(\mathbb{Z}[t,t^{-1}])$ on a product of locally infinite trees. They also ask whether the proof can be extended to show that $H_2(\mathbf{SL_2}(\mathbb{Z}[t,t^{-1}]);\mathbb{Z})$ is infinitely generated. Knudson proves that this is the case in [3] using algebraic methods.

Our goal is to prove the following closely related theorem:

Theorem 1 $H^2\left(\mathbf{SL_2}\left(\mathbb{Z}[t,t^{-1}]\right);\mathbb{Q}\right)$ *is infinite-dimensional*

The methods used will be geometric. We will define two spaces on which Γ acts, one a Euclidean building and the other a classifying space for Γ. A map between these spaces will allow us to define a family of independent cocycles in $H^2(\mathbf{SL_2}(\mathbb{Z}[t,t^{-1}]);\mathbb{Q})$.

The methods used are based on those of Cesa-Kelly in [4], where they are used to show that certain congruence subgroups of $\mathbf{SL_n}(\mathbb{Z}[t])$ have infinite-dimensional cohomology by using the action of that group on a building.

CHAPTER 2

THE EUCLIDEAN BUILDING

We begin by recalling the structure of a Euclidean building that acts on Γ. The following construction uses the notation of Bux-Wortman in [2]. Let ν_∞ and ν_0 be the valuations on $\mathbb{Q}(t)$ giving multiplicity of zeros at infinity and at zero, respectively. More precisely, $\nu_\infty\left(\frac{r(t)}{s(t)}\right) = \deg(s(t)) - \deg(r(t))$, and $\nu_0\left(\frac{r(t)}{s(t)}t^n\right) = n$, where t does not divide the polynomials r and s. Let T_∞ and T_0 be the Bruhat-Tits trees associated to $\mathbf{SL_2}(\mathbb{Q}(t))$ with the valuations ν_∞ and ν_0, respectively. Let $X = T_\infty \times T_0$.

Since $\mathbb{Q}((t^{-1}))$ (respectively $\mathbb{Q}((t)))$ is the completion of $\mathbb{Q}(t)$ with respect to ν_∞ (resp. ν_0), $\mathbf{SL_2}(\mathbb{Q}((t^{-1})))$ (resp. $\mathbf{SL_2}(\mathbb{Q}(t)))$ acts on the tree T_∞ (resp. T_0). Therefore the group $\mathbf{SL_2}(\mathbb{Q}((t^{-1}))) \times \mathbf{SL_2}(\mathbb{Q}((t)))$ acts on X. Thus $\mathbf{SL_2}(\mathbb{Q}(t))$ acts on X via its diagonal embedding into $\mathbf{SL_2}(\mathbb{Q}((t^{-1}))) \times \mathbf{SL_2}(\mathbb{Q}((t)))$, as does $\Gamma < \mathbf{SL_2}(\mathbb{Q}(t))$.

Let L_∞ (resp. L_0) be the unique geodesic line in T_∞ (resp. T_0) stabilized by the diagonal subgroup of $\mathbf{SL_2}(\mathbb{Q}(t))$. Let $\ell_\infty : \mathbb{R} \to L_\infty$ (resp. $\ell_0 : \mathbb{R} \to L_0$) be an isometry with $\ell_\infty(0)$ (resp. $\ell_0(0)$) the unique vertex stabilized by $\mathbf{SL_2}(\mathbb{Q}[t^{-1}])$ (resp. $\mathbf{SL_2}(\mathbb{Q}[t])$). Let $x_0 = (\ell_\infty(0), \ell_0(0))$ serve as a basepoint of X and $\Sigma = L_\infty \times L_0$ so that Σ is an apartment of X.

CHAPTER 3

THE ACTION OF Γ ON X

The goal of this section is to establish certain large-scale features of the action of Γ on X. In particular, we will find a horoball containing a sequence of points far from x_0 and show that the Γ-translates of this horoball are disjoint. The techniques here are similar to those used by Bux-Wortman in [5].

For certain parts of the proof, it will be convenient to work with $\Gamma_{\mathbb{Q}} = \mathbf{SL_2}(\mathbb{Q}[t, t^{-1}])$ instead of with Γ. Note that $\Gamma_{\mathbb{Q}}$ also acts on T_∞ and on T_0 and therefore acts diagonally on X.

We will also work with stabilizers of horoballs. Define

$$P_S = \left\{ \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \middle| \, a, b \in \mathbb{Q}((t^{-1})), c, d \in \mathbb{Q}((t)) \right\} \tag{3.1}$$

So that $P_S < \mathbf{SL_2}(\mathbb{Q}((t^{-1}))) \times \mathbf{SL_2}(\mathbb{Q}((t)))$ stabilizes the horoball in $T_\infty \times T_0$ in which we are interested.

We will also find it useful to define two other subgroups:

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a, b \in \mathbb{Q}(t) \right\} \tag{3.2}$$

and

$$U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| \, x \in \mathbb{Q}[t, t^{-1}] \right\} \tag{3.3}$$

We will refer also to $U_\Gamma = U \cap \Gamma$. All of these subgroups embed diagonally into P_S.

Lemma 2 *The double coset space $\Gamma_{\mathbb{Q}} \backslash \mathbf{SL_2}(\mathbb{Q}(t))/P$ is a single point.*

Proof. Note first that $\mathbf{SL_2}(\mathbb{Q}(t))$ acts on $\mathbb{P}^1(\mathbb{Q}((t)))$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$. Under this action, P is the stabilizer of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $\mathbf{SL_2}(\mathbb{Q}(t))/P = \mathbb{P}^1(\mathbb{Q}(t))$. Since $\Gamma_{\mathbb{Q}}$ acts transitively on $\mathbb{P}^1(\mathbb{Q}(t))$, $\Gamma_{\mathbb{Q}} \backslash \mathbf{SL_2}(\mathbb{Q}(t))/P$ is a single point. ■

Let $\rho : \mathbb{R}_{\geq 0} \to X$ be the geodesic ray with $\rho(s) = (\ell_\infty(s), \ell_0(s))$. Let β_ρ be the Busemann function for ρ with $\beta_\rho(x_0) = 0$. Thus $\beta_\rho^{-1}(\mathbb{R}_{\geq d})$ is a horoball based at $\rho(\infty)$, which is stabilized by P_S.

Let $x_n = (\ell_\infty(n), \ell_0(n))$.

Lemma 3 *For every $R > 0$,*

$$\beta_\rho^{-1}(R) \subset \mathrm{Nbhd}_{2+\frac{\sqrt{2}}{2}}(Px_m) \tag{3.4}$$

where m is an integer such that $|R - m| \leq \frac{1}{2}$.

Proof. Let $y = (\ell_\infty(R), \ell_0(R)) = \beta_\rho^{-1}(R) \cap \rho$ and $D = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.

Since $D^n \cdot (\ell_\infty(R), \ell_0(R)) = (\ell_\infty(R + 2n), \ell_0(R - 2n))$, we see that

$$\beta_\rho^{-1}(R) \cap \Sigma \subset \mathrm{Nbhd}_{2\sqrt{2}}(\{D^n\} \cdot y) \tag{3.5}$$

Since $X = U\Sigma$ and U preserves $\beta_\rho^{-1}(R)$,

$$\beta_\rho^{-1}(R) \subset \mathrm{Nbhd}_{\sqrt{2}}(U\{D^n\} \cdot y) \subset \mathrm{Nbhd}_{\sqrt{2}}(P \cdot y) \tag{3.6}$$

Since $d(x_m, y) < \frac{\sqrt{2}}{2}$, this gives the desired inclusion

$$\beta_\rho^{-1}(R) \subset \mathrm{Nbhd}_{\sqrt{2}+\frac{\sqrt{2}}{2}}(P \cdot x_m) \tag{3.7}$$

■

Lemma 4 *For every $C > 0$, there is some $N \in \mathbb{N}$ such that $d(x_n, \Gamma x_0) > C$ for every $n > N$.*

Proof. The following proof is closely based on the proof of Lemma 2.2 by Bux-Wortman in [2].

We will show that any subsequence of $\{x_{2n}\}$ is unbounded in the quotient space $\Gamma\backslash X$. This implies that only a finite number of points of $\{x_{2n}\}$ can be contained in any neighborhood of Γx_0. Let $C \in \mathbb{R}$. Since a finite number of points of $\{x_{2n}\}$ are contained in $\mathrm{Nbhd}_{C+\sqrt{2}}(\Gamma x_0)$ and $d(x_{2n}, x_{2n+1}) = \sqrt{2}$, a finite number of points of $\{x_{2n+1}\}$ are contained in $\mathrm{Nbhd}_C(\Gamma x_0)$. This suffices to prove the lemma.

The group $\mathbf{SL_2}(\mathbb{Q}(t)) \times \mathbf{SL_2}(\mathbb{Q}(t))$ acts componentwise on X, and has a metric induced by the valuations v_∞ and v_0. Under this metric the stabilizer of x_0 is a bounded subgroup. Thus, to prove that a set of vertices in $\Gamma\backslash X$ is not bounded, it suffices to prove that it has unbounded preimage under the projection

$$\Gamma\backslash\mathbf{SL_2}(\mathbb{Q}(t)) \times \mathbf{SL_2}(\mathbb{Q}(t)) \to \Gamma\backslash X \tag{3.8}$$

given by $\Gamma g \mapsto \Gamma g x_0$, where Γ is embedded diagonally into $\mathbf{SL_2}(\mathbb{Q}(t)) \times \mathbf{SL_2}(\mathbb{Q}(t))$.

Let $D = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ and $A = (D, D^{-1}) \in \mathbf{SL_2}(\mathbb{Q}(t)) \times \mathbf{SL_2}(\mathbb{Q}(t))$. Then $A^n x_0 = x_{2n}$. It therefore suffices to prove that any infinite subset of $\{\Gamma A^n\}$ is unbounded in $\Gamma \backslash \mathbf{SL_2}(\mathbb{Q}(t)) \times \mathbf{SL_2}(\mathbb{Q}(t))$.

Assume that this is not the case. That is, assume that there is some infinite subset $I \subset \mathbb{N}$ such that $\{\Gamma A^i\}_{i \in I}$ is bounded. Then there is some C such that for every $i \in I$ there is some $M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \Gamma$ such that the values of v_∞ of the coefficients of $M_i D^i$ are bounded from below by C and the values of v_0 of the coefficients of $M_i D^{-i}$ are also bounded from below by C. Then

$$C \leq v_\infty(a_n t^n) = v_\infty(a_n) + n v_\infty(t) = v_\infty(a_n) - n \tag{3.9}$$

and

$$C \leq v_0(a_n t^{-n}) = v_0(a_n) - n v_0(t) = v_0(a_n) - n \tag{3.10}$$

This gives that $v_\infty(a_n) \geq 1$ and $v_0(a_n) \geq 1$ whenever $n \geq 1 - C$, which implies that $a_n = 0$.

The same argument shows that $c_n = 0$, which implies that M_i is not in Γ when $i \geq 1 - C$. ■

Lemma 5 *There is some $R > 0$ such that $\Gamma_{\mathbb{Q}} \cdot x_0 \cap \beta_\rho^{-1}([R, \infty)) = \emptyset$.*

Proof. Since $\beta_\rho^{-1}([R, \infty))$ is a union of the horospheres $\beta_\rho^{-1}(r)$ for $r \in [R, \infty)$, Lemma 3 guarantees that

$$\beta_\rho^{-1}([R, \infty)) \subset \bigcup_{\{m \in \mathbb{Z} | m > R - \frac{1}{2}\}} \mathrm{Nbhd}_{\sqrt{2} + \frac{\sqrt{2}}{2}}(P \cdot x_m) \tag{3.11}$$

Recall that $d(x_m, \Gamma_{\mathbb{Q}} \cdot x_0) > \sqrt{2} + \frac{\sqrt{2}}{2}$ for large values of R. This implies that $d(\Gamma_{\mathbb{Q}} \cdot x_m, \Gamma_{\mathbb{Q}} \cdot x_0) > \sqrt{2} + \frac{\sqrt{2}}{2}$. Thus, we may conclude that $\Gamma_{\mathbb{Q}} \cdot x_0 \cap \beta_\rho^{-1}([R, \infty)) = \emptyset$. ■

Fix R as in Lemma 5. Let d be the maximum distance from a point in the horosphere $\beta_\rho^{-1}(R)$ to the orbit $\Gamma_{\mathbb{Q}} \cdot x_0$.

Lemma 6 *d is finite.*

Proof. By Lemma 3, $\beta_\rho^{-1}(R) \subset \mathrm{Nbhd}_r(P x_n)$ for some $n > 0$, where $r = 2 + \frac{\sqrt{2}}{2}$. Thus for every $x \in \beta_\rho^{-1}(R)$ there is some $p \in P$ such that $d(x, p \cdot x_n) < r$. Since $d(p \cdot x_n, p \cdot x_0) = d(x_n, x_0) = n\sqrt{2}$, we have $d(x, p \cdot x_0) \leq r + n\sqrt{2}$. Since this bound is independent of x, it follows that $d \leq r + n\sqrt{2}$. ■

Let $H = \beta_\rho^{-1}([R + d, \infty))$.

Lemma 7 *Let $\gamma \in \Gamma_{\mathbb{Q}}$. If $\gamma H \cap H \neq \emptyset$, then $\gamma H = H$ and $\gamma \in P_S \cap \Gamma_{\mathbb{Q}}$.*

Proof. Let $\Sigma \subset X$ be an apartment whose boundary contains the chamber C of ∂X containing the point $\rho(\infty)$ as well as the chamber γC. The endpoints of the arc C are $\ell_\infty(\infty)$ and $\ell_0(\infty)$. Thus, any chamber adjacent to C contains one of these two points. Because $\mathrm{Stab}_\Gamma(\ell_\infty(\infty)) = \mathrm{Stab}_\Gamma(\ell_0(\infty)) < P$, any element of $\Gamma_{\mathbb{Q}}$ either stabilizes both $\ell_\infty(\infty)$ and $\ell_0(\infty)$ or neither. Therefore, C and γC cannot be adjacent. Since each apartment of ∂X contains exactly four chambers, the two chambers are either equal or opposite.

If C and γC are opposite, then $\Sigma \cap H \cap \gamma H$ is contained in a neighborhood of a hyperplane in Σ. Suppose $x \in \Sigma \cap \gamma H \cap \beta_\rho^{-1}(R)$. It follows from the choice of d that there is some $y \in \Gamma_{\mathbb{Q}} \cdot x_0$ such that $d(x, y) \leq d$. Since $\beta_{\gamma\rho}(x) \geq R + d$, it is clear that $\beta_{\gamma\rho}(y) \geq R$. Therefore $\beta_\rho(\gamma^{-1} y) \geq R$, which contradicts Lemma 5, since $\gamma^{-1} y \in \Gamma_{\mathbb{Q}} \cdot x_0$. Thus $\gamma C = C$, which implies that $\gamma \in P_S$. It follows that H and γH are horoballs based at the same boundary point. Since γ preserves distance from $\Gamma_{\mathbb{Q}} \cdot x_0$, $\gamma H = H$. ■

CHAPTER 4

THE CLASSIFYING SPACE

We will now construct a second space on which Γ acts geometrically and freely. A map between the two spaces will allow us to think about the cohomology of Γ in the familiar context of a product of trees.

Let X_0 be a discrete collection of points $\{x_\gamma | \gamma \in \Gamma\}$. Γ acts freely on this set by $\gamma' \cdot x_\gamma = x_{\gamma'\gamma}$. Define a Γ-equivariant map $\psi_0 : X_0 \to X$ by $\psi_0(x_\gamma) = \gamma \cdot x_0$. By Lemma 5, $\psi_0(X_0) \cap H = \emptyset$. Note also that $\psi_0(X_0) \cap \gamma H = \emptyset$ for every $\gamma \in \Gamma$.

Construct X_1 from X_0 by attaching a 1-cell $F_{x_\gamma, x_{\gamma'}}$ of length 1 between a pair of points $x_\gamma, x_{\gamma'} \in X_0$. For $\zeta \in \Gamma$, let $\zeta : F_{x_\gamma, x_{\gamma'}} \to F_{x_{\zeta\gamma}, x_{\zeta\gamma'}}$ be the unique distance-preserving map with $\zeta(x_\gamma) = \zeta(x_\gamma)$ and $\zeta(x_{\gamma'}) = \zeta(x_{\gamma'})$. This defines a Γ-action on X_1 that extends the Γ-action on X_0.

We wish to define a Γ-equivariant map $\psi_1 : X_1 \to X$ extending ψ_0 and with

$$\psi_1(X_1) \subset X - \bigcup_{\gamma \in \Gamma} \gamma H \tag{4.1}$$

For every nonidentity element $\gamma \in \Gamma$, choose a path c_γ from x_0 to γx_0. Since $\partial(\gamma H)$ is connected for all $\gamma \in \Gamma$, we can choose all such paths to lie outside of $\bigcup_{\gamma \in \Gamma} \gamma H$. We may then define $\psi_1(F_{x_\gamma, x_{\gamma'}})$ to lie along the path $\gamma c_{(\gamma^{-1}\gamma')}$.

We will build a family of spaces X_n inductively. Beginning with an n-dimensional, $(n-1)$-connected cell complex X_n on which Γ acts freely and a Γ-equivariant map $\psi_n : X_n \to X$, we may construct an $n+1$-dimensional, n-connected cell complex X_{n+1} on which Γ acts freely and a Γ-equivariant map $\psi_{n+1} : X_{n+1} \to X$. For every map $s : S^n \to X_n$, attach an $(n+1)$-cell F_s along γs for every $\gamma \in \Gamma$. This results in an n-connected space X_{n+1}. Define a Γ-action on X_{n+1} by $\gamma F_s = F_{\gamma s}$. For each s as above, define $\psi_{n+1,s} : F_s \to X$ to extend $\psi_n(\partial F_s)$ with $\psi_{n+1,\gamma s} = \gamma \psi_{n+1,s}$. Because $\pi_{n+1}(X)$ is trivial for all n, this is always possible. There is a unique Γ-equivariant map ψ_{n+1} extending these maps.

Since the focus of Theorem 1 is second-dimensional cohomology, we will use the space X_3 extensively. Therefore we will let $\psi = \psi_3$. Since $\Gamma\backslash X_3$ is the 3-skeleton of a $K(\Gamma, 1)$, $H^2(\Gamma; \mathbb{Q}) = H^2(X_3; \mathbb{Q})$.

CHAPTER 5

LOCAL COHOMOLOGY

Before defining cocyles on $\Gamma \backslash X_3$, we will define cocycles in the relative homology groups of several subspaces of X.

Recall that $x_n = (\ell_\infty(n), \ell_0(n))$. We will assume from now on that $n > R + d$ so that $x_n \in H$. Recall also

$$U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{Q}[t, t^{-1}] \right\} \tag{5.1}$$

We will regard U as a diagonally embedded subgroup of $\Gamma_\mathbb{Q} \times \Gamma_\mathbb{Q}$.

We define two submodules of $\mathbb{Q}[t, t^{-1}]$:

$$M_n = \left\{ x \in \mathbb{Q}[t, t^{-1}] \middle| x = \sum_{i=-n}^{n} a_i t^i \right\} \tag{5.2}$$

$$M^n = \left\{ x \in \mathbb{Q}[t, t^{-1}] \middle| x = \sum_{i=-k}^{-n-1} a_i t^i + \sum_{i=n+1}^{k} a_i t^i \right\} \tag{5.3}$$

We also define corresponding normal subgroups of U:

$$U_n = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in M_n \right\} \tag{5.4}$$

$$U^n = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in M^n \right\} \tag{5.5}$$

Note that $U = U_n \times U^n$. Since the elements of M_n are precisely those elements x of $\mathbb{Q}[t, t^{-1}]$ with $\nu_0 \geq -n$ and $\nu_\infty \geq -n$, U_n is the stabilizer in U of x_n.

Let S_n be the star of x_n in X (that is, the collection of cells having x_n as a vertex) and C the cell in S_n containing x_{n-1}. Let $S_n^\downarrow = U_n C$. We will define cocycles $\varphi_n \in H^2(S_n^\downarrow, S_n^\downarrow \cap \partial S_n; \mathbb{Q})$. Averaging these over cosets of U_Γ in Γ will give us cocycles in $H^2\left(\mathbf{SL_2}\left(\mathbb{Z}[t, t^{-1}]\right); \mathbb{Q}\right)$, which we will use to prove Theorem 1.

Lemma 8 $\left\{ \begin{pmatrix} 1 & at^{-n} + bt^n \\ 0 & 1 \end{pmatrix} \middle| a, b \in \mathbb{Q} \right\} < U_n$ *acts transitively on the set of 2-cells in* $S_n^\downarrow$.

Proof. Since U_n fixes x_n and preserves the Busemann function, it will act on $S_n^{\downarrow}$. In particular, we will find subsets of U_n that act transitively on the sets

$$E = \{\ell_\infty(n) \times e | e \subset T_0 \text{ an edge}, \ell_0(n) \in e, \ell_0(n+1) \notin e\} \tag{5.6}$$

and

$$F = \{e \times \ell_0(n) | e \subset T_\infty \text{ an edge}, \ell_\infty(n) \in e, \ell_\infty(n+1) \notin e\} \tag{5.7}$$

Let

$$U_n^E = \left\{ \begin{pmatrix} 1 & at^{-n} \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{Q} \right\} \tag{5.8}$$

and

$$U_n^F = \left\{ \begin{pmatrix} 1 & bt^{n} \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{Q} \right\} \tag{5.9}$$

U_n^E acts transitively on the set of edges in T_0 incident on $\ell_0(n)$ and not on $\ell_0(n+1)$. Since U_n^E stabilizes x_n, it acts transitively on E. Denote by e_0 the edge in E between x_n and $(\ell_\infty(n), \ell_0(n-1))$. Let $e_a = \begin{pmatrix} 1 & at^{-n} \\ 0 & 1 \end{pmatrix} e_0$ for each $a \in \mathbb{Q}$.

U_n^F acts transitively on the set of edges in T_∞ incident on $\ell_\infty(n)$ and not on $\ell_\infty(n+1)$. Since U_n^F stabilizes x_n, it acts transitively on F. Denote by f_0 the edge in F between x_n and $(\ell_\infty(n-1), \ell_0(n))$. Let $f_b = \begin{pmatrix} 1 & bt^{n} \\ 0 & 1 \end{pmatrix} f_0$ for each $b \in \mathbb{Q}$.

Since any 2-cell in $S_n^{\downarrow}$ contains a unique pair of edges e_a and f_b in its boundary,

$$U_n^E \times U_n^F = \left\{ \begin{pmatrix} 1 & at^{-n} + bt^{n} \\ 0 & 1 \end{pmatrix} \middle| a, b \in \mathbb{Q} \right\} \tag{5.10}$$

acts transitively on the set of 2-cells in $S_n^{\downarrow}$ as well as on E and F. ■

Let $C_{0,0}$ be the 2-cell in $S_n^{\downarrow}$ containing x_{n-1}, and let $C_{a,b} = \begin{pmatrix} 1 & at^{-n} + bt^{n} \\ 0 & 1 \end{pmatrix} C_{0,0}$. Orient each edge e_a and f_b with initial vertex x_n. We may then choose an orientation for each $C_{a,b}$ such that we have $\partial C_{a,b} = e_a - f_b + D$ for some chain $D \subset S_n^{\downarrow} \cap \partial S_n$.

Define a cochain φ_n by $\varphi_n(C_{a,b}) = ab$. Since X is 2-dimensional, any 2-cochain is a cocycle.

We will define cocycles in $H^2(\Gamma \backslash X_3; \mathbb{Q})$ as sums of the φ_n's. The following lemma will be used in proving that these cocycles are well-defined.

Lemma 9 *φ_n is U_n-invarant.*

Proof. Let a *basic cycle* in $S_n^{\downarrow}$ be one of the form $C_{x,y} - C_{x',y} - C_{x,y'} + C_{x',y'}$. We will show first that φ_n is U_n-invariant on basic cycles, then show that any cycle is a sum of basic cycles.

Let

$$u = \begin{pmatrix} 1 & \sum_{i=-n}^{n} a_i t^i \\ 0 & 1 \end{pmatrix} \in U_n \tag{5.11}$$

and D be a basic cycle.

$$\begin{aligned}
\varphi_n(uD) &= \varphi_n(u(C_{x,y} - C_{x',y} - C_{x,y'} + C_{x',y'})) && (5.12)\\
&= \varphi_n(uC_{x,y} - uC_{x',y} - uC_{x,y'} + uC_{x',y'}) && (5.13)\\
&= \varphi_n(C_{x+a_{-n},y+a_n} - C_{x'+a_{-n},y+a_n} - C_{x+a_{-n},y'+a_n} + C_{x'+a_{-n},y'+a_n}) && (5.14)\\
&= (x + a_{-n})(y + a_n) - (x' + a_{-n})(y + a_n) - (x + a_{-n})(y' + a_n) && (5.15)\\
&\qquad + (x' + a_{-n})(y' + a_n) && (5.16)\\
&= xy + x'y' - xy' - x'y && (5.17)\\
&= \varphi_n(D) && (5.18)
\end{aligned}$$

Thus φ_n is U_n-invariant on basic cycles.

It remains to show that all cycles are sums of basic cycles. Define the length l of a chain in $H_2(S_n^{\downarrow}, S_n^{\downarrow} \cap \partial S_n)$ by

$$l\left(\sum_{i,j\in\mathbb{Z}} \alpha_{i,j} C_{i,j}\right) = \sum_{i,j\in\mathbb{Z}} |\alpha_{i,j}| \tag{5.19}$$

Suppose there are cycles which are not sums of basic cycles. Let $B = \sum_{i,j\in\mathbb{Z}} \alpha_{i,j} C_{i,j}$ be such a cycle with the property that $l(B) \leq l(B')$, where B' is any other such cycle. Let $C_{x,y}$ be a 2-cell such that $\alpha_{x,y} > 0$. Since B is a cycle, $\partial B = 0$; therefore there are some $x', y' \in \mathbb{Z}$ such that $\alpha_{x',y}, \alpha_{x,y'} < 0$. Then

$$l(B - (C_{x,y} - C_{x',y} - C_{x,y'} + C_{x',y'})) \leq l(B) - 2 \tag{5.20}$$

Since this cycle differs from B by a basic cycle, it cannot be written as a sum of basic cycles, contradicting the assumption that B is the shortest cycle with that property. ■

CHAPTER 6

COHOMOLOGY

We now wish to define cocycles in $H^2(\Gamma\backslash X_3;\mathbb{Q})$ by averaging φ_n over cosets of U_Γ in Γ.

Let $Y_n = U_n\Sigma$. Since Y_n is a fundamental domain for the action of U^n on X, we may identify it with the quotient $U^n\backslash X$. Let $\theta_n : X \to Y_n$ be the quotient map. Note that θ_n is U_n-equivariant.

We may now define $\Phi_n \in H^2(\Gamma\backslash X_3;\mathbb{Q})$ by

$$\Phi_n(\Gamma D) = \sum_{gU_\Gamma \in \Gamma/U_\Gamma} \varphi_n(\theta_n(\psi(g^{-1}D)) \cap S_n^{\downarrow}) \tag{6.1}$$

for any 2-cell D in X_3. The next several lemmas show that Φ_n is well-defined, a cocycle, and U_n-invariant.

Lemma 10 *Φ_n is well-defined. In particular, it is independent of the choice of coset representatives in ΓD and gU_Γ.*

Proof. Let $u \in U_\Gamma, u^{-1} = u^n u_n$ for $u^n \in U^n \cap \Gamma$ and $u_n \in U_n \cap \Gamma$. Then

$$\begin{aligned}
\varphi_n(\theta_n(\psi((gu)^{-1}D)) \cap S_n^{\downarrow}) &= \varphi_n(\theta_n(\psi(u^{-1}g^{-1}D)) \cap S_n^{\downarrow}) && (6.2)\\
&= \varphi_n(\theta_n(u^{-1}\psi(g^{-1}D)) \cap S_n^{\downarrow}) && (6.3)\\
&= \varphi_n(\theta_n(u^n u_n\psi(g^{-1}D)) \cap S_n^{\downarrow}) && (6.4)\\
&= \varphi_n(\theta_n(u_n\psi(g^{-1}D)) \cap S_n^{\downarrow}) && (6.5)\\
&= \varphi_n(u_n\theta_n(\psi(g^{-1}D)) \cap S_n^{\downarrow}) && (6.6)\\
&= \varphi_n(\theta_n(\psi(g^{-1}D)) \cap S_n^{\downarrow}) && (6.7)
\end{aligned}$$

So Φ_n is independent of coset representative in gU_Γ.

Let $\gamma \in \Gamma$. Then

$$\Phi_n(\Gamma(\gamma D)) = \sum_{gU_\Gamma \in \Gamma/U_\Gamma} \varphi_n(\theta_n(\psi(g^{-1}\gamma D) \cap S_n^{\downarrow})) \tag{6.8}$$

$$= \sum_{gU_\Gamma \in \Gamma/U_\Gamma} \varphi_n(\theta_n(\psi((\gamma^{-1}g)^{-1}D) \cap S_n^{\downarrow})) \tag{6.9}$$

$$= \sum_{\gamma^{-1}gU_\Gamma \in \Gamma/U_\Gamma} \varphi_n(\theta_n(\psi((\gamma^{-1}g)^{-1}D) \cap S_n^{\downarrow})) \tag{6.10}$$

$$= \sum_{gU_\Gamma \in \Gamma/U_\Gamma} \varphi_n(\theta_n(\psi(g^{-1}D) \cap S_n^{\downarrow})) \tag{6.11}$$

$$= \Phi_n(\Gamma D) \tag{6.12}$$

Therefore Φ_n is also independent of coset representative in ΓD. ■

Lemma 11 Φ_n *is a cocycle.*

Proof. Let ΓD be a 3-cell in $\Gamma\backslash X_3$ corresponding to the 3-cell D in X_3. Then ∂D is a 2-sphere. Since X contains no nontrivial 2-spheres, $\psi(\partial D)$ is trivial. Thus,

$$\varphi_n(\theta_n(\psi(g^{-1}\partial D)) \cap S_n^{\downarrow}) = 0 \tag{6.13}$$

for any $g \in \Gamma$ and $\Phi_n(\Gamma D) = 0$.

■

Lemma 12 Φ_n *is* U_n*-invariant.*

Proof. It suffices to show that $\theta_n\psi(D) \cap S_n$ is supported on $S_n^{\downarrow}$ for every disk D in X_3 since every φ_n is U_n-invariant.

Let $H_n = \beta_\rho^{-1}([\beta_\rho(x_n), \infty))$. Since $H_n \subset H$, the definition of ψ implies that $\psi(X_1) \cap H_n = \emptyset$. Thus, $\partial\psi(D)$ is entirely outside of H_n. Since U^n preserves H_n, we may also conclude that $\partial\theta_n(\psi(D))$ is outside $\theta_n(H_n)$.

Suppose there is some cell $C_0 \subset \mathrm{Supp}(\theta_n(\psi(D))$ with $C_0 \subset S_n$ and $C_0 \not\subset S_n^{\downarrow}$. We will find an infinite family of cells which must also be contained in $\mathrm{Supp}(\theta_n(\psi(D))$, contradicting the compactness of $\theta_n(\psi(D))$.

Since $U_n\Sigma = Y_n$ and $S_n^{\downarrow}$ is U_n-invariant, we may assume without loss of generality that $C_0 \subset \Sigma$. Then C_0 is one of $\ell_\infty([n-1,n]) \times \ell_0([n,n+1])$, $\ell_\infty([n,n+1]) \times \ell_0([n-1,n])$, or $\ell_\infty([n,n+1]) \times \ell_0([n,n+1])$. We will examine in detail the case where $C_0 = \ell_\infty([n-1,n]) \times \ell_0([n,n+1])$. The others are parallel.

Let $C_i = \ell_\infty([n-1,n]) \times \ell_0([n+i,n+i+1])$ and $e_i = \ell_\infty([n-1,n]) \times \ell_0(n+i)$ so that $C_i, e_i \subset \Sigma$ with C_{i-1} and C_i adjacent along e_i. We will show by induction that all C_i are in $\mathrm{Supp}(\theta_n(\psi(D))$.

Since $e_i \subset H_n$, it is clear that $e_i \not\subset \partial\theta_n(\psi(D))$. This if $\mathrm{Supp}(\theta_n(\psi(D))$ contains C_{i-1}, it must also contain some other cell with e_i in its boundary. Such a cell must be of the form $\ell_\infty([n-1,n]) \times e_i^0$ for some edge $e_i^0 \subset T_0$ with $\ell_0(n+i)$ one endpoint. Since the action of U_n on T_0 fixes all edges incident to $\ell_0(n+i)$, the only cells of this form in $U_n\Sigma$ are the ones in Σ itself: C_{i-1} and C_i. Thus $C_i \subset \mathrm{Supp}(\theta_n(\psi(D))$.

This shows that $\mathrm{Supp}(\theta_n(\psi(D))$ contains infinitely many cells and is not compact. ■

Now that we have defined a family of cocyles, it remains to show that they are independent. We will do this by exhibiting a family of cycles $\widetilde{B_{2n}}$ such that $\Phi_{2n}(\Gamma\widetilde{B_{2n}}) = 1$ and $\Phi_k(\Gamma\widetilde{B_{2n}}) = 0$ for $k > 2n$.

Let F_{2n} be the traingle in Σ with vertices x_n, $(\ell_\infty(n), \ell_0(-n))$, and $(\ell_\infty(-n), \ell_0(n))$. Let

$$B_{2n} = F_{2n} - \begin{pmatrix} 1 & t^{2n} \\ 0 & 1 \end{pmatrix} F_{2n} - \begin{pmatrix} 1 & t^{-2n} \\ 0 & 1 \end{pmatrix} F_{2n} + \begin{pmatrix} 1 & t^{-2n} + t^{2n} \\ 0 & 1 \end{pmatrix} F_{2n}$$

Then B_{2n} is a square with $B_{2n} \cap S_{2n}^{\downarrow} = C_{0,0} - C_{1,0} - C_{0,1} + C_{1,1}$.

Lemma 13 *For every $n \in \mathbb{N}$, there is some disc $\widetilde{B_{2n}} \subset X_3$ such that $\psi(\widetilde{B_{2n}}) \cap H = B_{2n}$.*

Proof. Letting $u = \begin{pmatrix} 1 & t^{-2n} + t^{2n} \\ 0 & 1 \end{pmatrix}$, the vertices of B_{2n} are $A^n x_0$, $A^{-n} x_0$, $uA^n x_0$, and $uA^{-n} x_0$. In particular, these four vertices are in Γx_0 and therefore in $\psi(X_0)$. Let v_i and v_j be two vertices on an edge of the square, with $\widetilde{v_i} \in \psi^{-1}(v_i)$ and $\widetilde{v_j} \in \psi^{-1}(v_j)$. Since X_3 is connected, there is a path $\widetilde{c}$ between $\widetilde{v_i}$ and $\widetilde{v_j}$. Thus $c_{i,j} = \psi(\widetilde{c})$ is a path connecting v_i and v_j. By the definition of ψ, $c_{i,j}$ is disjoint from H. Repeating this for all four edges gives a path c in X which bounds a disk B'_{2n} such that $B_{2n} \cap H = B'_{2n} \cap H$ and is the image of a path $\widetilde{c}$ in X_3. Since X_3 is 2-connected, there is a disk $\widetilde{B_{2n}}$ that fills $\widetilde{c}$, and $\psi(\widetilde{B_{2n}}) = B'_{2n}$. ■

Lemma 14 $\Phi_{2n}(\Gamma\widetilde{B_n}) = 1$ *and* $\Phi_k(\Gamma\widetilde{B_{2n}}) = 0$ *for* $k > 2n$.

Proof. For $2n > R + d$, $S_{2n}^{\downarrow} \subset H$. Since $P_S \cap \Gamma$, the stabilizer of H in Γ, preserves distance from the horoball $\beta_\rho^{-1}(0)$, $\varphi_k(pB_{2n}) = 0$ for $p \in P_S \cap \Gamma$. For any $\gamma \in \Gamma - P_S$, we know $\gamma H \cap H = \emptyset$, so $\varphi_k(\gamma B_{2n}) = 0$ for any k and n. Thus $\Phi_k(\Gamma\widetilde{B_{2n}}) = 0$ for $k > 2n$.

By the definition of φ_n, $\varphi_{2n}(B_{2n}) = 1$. As above, $\varphi_{2n}(\gamma B_{2n}) = 0$ for any $\gamma \in \Gamma - U_\Gamma$. Therefore $\Phi_{2n}(\Gamma\widetilde{B_{2n}}) = 1$. ■

Thus, for any k the set of cocycles $\{\Phi_{R+d+2}, \Phi_{R+d+4}, \ldots \Phi_{R+d+2k}\}$ is independent, which suffices to prove Theorem 1.

REFERENCES

[1] S. Krstić and J. McCool, "The non-finite presentability of $IA(F_3)$ and $GL_2(\mathbf{Z}[t,t^{-1}])$." *Inventiones Mathematicae*, vol. 129, pp. 595–606, 1997.

[2] K.-U. Bux and K. Wortman, "A Geometric proof that $\mathbf{SL_2}(\mathbb{Z}[t,t^{-1}])$ is not finitely presented," *Algebr. Geom. Topol. 6*, vol. 167, pp. 839–852, 2006.

[3] K. Knudson, "Homology and finiteness properties of $SL_2(\mathbb{Z}[t,t^{-1}])$," *preprint*, 2008.

[4] M. Cesa and B. Kelly, "Congruence subgroups of $\mathbf{SL_n}(\mathbb{Z}[t])$ with infinite dimensional cohomology," *preprint*, 2014.

[5] K.-U. Bux and K. Wortman, "Connectivity properties of horospheres in euclidean buildings and applications to finiteness properties of discrete groups," *Inventiones Mathematicae*, vol. 185, pp. 395–419, 2011.